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Examples of Heun and Mathieu functions as solutions of wave equations in curved spaces

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Abstract

We give examples of where the Heun function exists as solutions of wave equations encountered in general relativity. As a new example we find that while the Dirac equation written in the background of Nutku helicoid metric yields Mathieu functions as its solutions in four spacetime dimensions, the trivial generalization to five dimensions results in the double confluent Heun function. We reduce this solution to the Mathieu function with some transformations.

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1. Introduction

Most of the theoretical physics known today is described by a rather small number of differential equations. If we study only linear problems, the different special cases of the hypergeometric or the confluent hypergeometric equation often suffice to analyse scores of different phenomena. These are two equations of the Fuchsian type with three regular singular points and one regular, one irregular singular point respectively. Both of these equations have simple recursion relations between two consecutive coefficients when a power expansion solution is attempted. This fact gives, in many instances, sufficient information on the general behaviour of the solution. If the problem is nonlinear, one can usually fit the equation describing the process to one of the different forms of the Painlevé equations [1].

It seems to be a miracle that such diverse phenomena, with examples in potential theory or wave equations with physical applications, can be described with so few equations. Physicists are lucky since most of the phenomena in physics of the present can be described in terms of these rather simple functions. Perhaps this refers to a symmetry beyond all these things, like the occurrence of hypergeometric functions may signal the presence of conformal symmetry.

In the linear case, sometimes it is necessary to go to equations with more singular points. The Heun equation [2–4], its confluent cases, or its special cases, Mathieu, Lamé, Coulomb spheroidal equations, etc all have additional singular points, either four regular or two regular, one irregular or two irregular. The price you pay is the fact that in general there exists no recursion relation between two consecutive coefficients when a power series expansion is used for the solution. A three or four way recursion relation is often awkward, and it is not easy to deduce valuable analytical information from such an expansion. New versions of the computer package Maple, for example Maple 10, give graphical representations of Heun functions. Although this is a great help, it is sometimes awkward to use these functions as potentials in either wave or Schroedinger equations. This may be a reason why much less is known about these equations compared to hypergeometric functions and all the other functions derived from them.

One encounters Mathieu functions when one uses elliptic coordinates, instead of circular ones, even in two dimensions [5]. Phenomena described by Heun equations are not uncommon when one studies problems in atomic physics with certain potentials [6] which combine different inverse powers starting from the first up to fourth power or combining the quadratic potential with inverse powers of two, four, six, etc. They also arise when one studies symmetric double Morse potentials. Slavyanov and Lay [7] describe different physical applications of these equations. Atomic physics problems such as separated double wells, Stark effect [7, 8], hydrogen–molecule ion [7, 9] use these forms of these equations. Many different problems in solid state physics such as dislocation movement in crystalline materials, quantum diffusion of kinks along dislocations are also solved in terms of these functions [7]. The famous Hill equation [7, 10] for lunar perigee can be cited for an early application in celestial mechanics.

In general relativity, while solving wave equations, we also encounter different forms of the Heun equations. Teukolsky [7, 11] studied the perturbations of the Kerr metric and found out that they were described by two coupled singly confluent Heun equations. Quasi-normal modes of rotational gravitational singularities were also studied by solving this system of equations [7, 12].

In recent applications they become indispensable when one studies phenomena in higher dimensions, for example the article by G Siopsis [13], or phenomena using different geometries. An example of the latter case is seen in the example of wave equations written in the background of these metrics. For instance, in four dimensions, we may write wave equations in the background of 4D Euclidean gravity solutions. For the metric written in the Eguchi–Hanson instanton [14] background, the hypergeometric function is sufficient to describe the spinor field solutions [15, 16]. One, however, has to use Mathieu functions to describe even the scalar field in the background of the Nutku helicoid instanton [17, 18] when the separation of variables method is used for the solution. Schmid *et al* [19] have written a short note describing the occurrence of these equations in general relativity. Their examples are the Dirac equation in the Kerr–Newman metric and static perturbations of the non-extremal Reisner–Nordström solution. They encounter the generalized Heun equation [3, 4, 20] while looking for the solutions in these metrics. Here we see that as the metric becomes more complicated, one has to solve equations with a larger number of singular points, with no simple recursion relations if one attempts a series-type solution. As a particular case of confluent Heun equation, Fiziev studied the exact solutions of the Regge–Wheeler equation [21, 22]. One also sees that if one studies similar phenomena in higher dimensions, unless the metric is a product of simple ones, one has higher chances of encountering Heun-type equations as in the references given [23, 24].

Here we want to give further examples to this general behaviour. Our first example will be the case already studied by Sucu and Ünal [15]. In their paper, they studied the spinor field in the background of the Nutku helicoid instanton. They obtained an exact solution, which, however, can be expanded in terms of Mathieu functions [25]. At this point, note that their solution in the background of the Eguchi–Hanson instanton is expanded in terms of hypergeometric functions. Taking the next 4D Euclidean gravity solution in row, the Nutku helicoid instanton with two centres [26, 27], results in a function with a higher singularity structure. The second example is the similar equation in five dimensions. Here, the metric is extended to five dimensions with a simple addition of the time coordinate. In this case we obtain the double confluent Heun function as the solution, which can be reduced to the Mathieu function with a variable transformation.

At this point, the motive for our paper becomes clear. Both in four and five dimensions, in the background metric of the Nutku helicoid instanton, we can write the solutions of the Dirac equations in terms of Mathieu functions. There is a catch here, though. Although we can express the solution in a closed form in the four-dimensional case, as done by Sucu and Ünal [15], this is not possible using the solutions in five dimensions, since the constants used in the two equations are not the same.

In four dimensions, we can also calculate the Greens function for this differential equation in a closed form following the steps in [18]. In five dimensions, we could not succeed in summing the infinite series of the product of Mathieu functions to express both the solution to the differential equation and the Greens function of the same differential equation in a closed form. This is due to the existence of two different constants in the two Mathieu functions used in the expansion. What we wanted to show is that going to one higher dimension, we got solutions which were more complicated.

We describe our examples in the following sections, first the case in four, then in five dimensions. We then give the solutions for the scalar operator. We end with some additional remarks. In our work we use only the massless field, since taking the massive field is technically like going one higher dimension, which complicates the problem. We instead go to one higher dimension in an explicit fashion in the following section.

2. Equations in four dimensions

The Nutku helicoid metric is given as

$$\mathrm{d}s^2 = \frac{1}{\sqrt{1+\frac{a^2}{r^2}}}\left[\mathrm{d}r^2 + (r^2+a^2)\,\mathrm{d}\theta^2 + \left(1+\frac{a^2}{r^2}\sin^2\theta\right)\mathrm{d}y^2 - \frac{a^2}{r^2}\sin 2\theta\,\mathrm{d}y\,\mathrm{d}z + \left(1+\frac{a^2}{r^2}\cos^2\theta\right)\mathrm{d}z^2\right], \tag{1}$$

where $0 < r < \infty$, $0 \leqslant \theta \leqslant 2\pi$, y and z are along the Killing directions and will be taken to be periodic coordinates on a 2-torus [18]. This is an example of a multi-centre metric. This metric reduces to the flat metric if we take $a = 0$,

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{2}$$

If we make the following transformation:

$$r = a\sinh x, \tag{3}$$

the metric is written as

$$\mathrm{d}s^2 = \frac{a^2}{2}\sinh 2x\,(\mathrm{d}x^2+\mathrm{d}\theta^2) + \frac{2}{\sinh 2x}[(\sinh^2 x + \sin^2\theta)\,\mathrm{d}y^2 - \sin 2\theta\,\mathrm{d}y\,\mathrm{d}z + (\sinh^2 x+\cos^2\theta)\,\mathrm{d}z^2]. \tag{4}$$

We use the NP formalism [28, 29] in four Euclidean dimensions [30–32]. To write the Dirac equation in this formalism, we need to choose the base vectors and calculate the spin coefficients, the differential operators and the γ matrices in curved space. We take base vectors

$$e_a^\mu = \{l^\mu, \bar{l}^\mu, m^\mu, \bar{m}^\mu\}, \tag{5}$$

to give

$$\mathrm{d}s^2 = l \otimes \bar{l} + \bar{l} \otimes l + m \otimes \bar{m} + \bar{m} \otimes m. \tag{6}$$

The tetrad

$$e^a = e^a_\nu \,\mathrm{d}x^\nu, \tag{7}$$

satisfies

$$\eta_{ab} = e_a^\mu e_b^\nu g_{\mu\nu}, \tag{8}$$

where η_{ab} is the flat metric. We choose

$$l^\mu = \frac{1}{a\sqrt{\sinh 2x}}(1, \mathrm{i}, 0, 0), \tag{9}$$

$$m^\mu = \frac{1}{\sqrt{\sinh 2x}}(0, 0, \cosh(x - \mathrm{i}\theta), \mathrm{i}\sinh(x - \mathrm{i}\theta)), \tag{10}$$

giving the two non-zero spin coefficients

$$\epsilon = \bar{\epsilon} = \frac{\cosh(2x)}{a \sinh^{3/2} 2x}, \tag{11}$$

$$\sigma = \bar{\sigma} = \frac{2}{a \sinh^{3/2} 2x}. \tag{12}$$

The rest of the spin coefficients,

$$\kappa = \nu = \gamma = \alpha = \beta = \pi = \tau = \mu = \lambda = \rho = 0. \tag{13}$$

These expressions give the differential operators

$$D = m^\mu \partial_\mu = \frac{1}{\sqrt{\sinh 2x}}[\cosh(x - \mathrm{i}\theta)\partial_y + \mathrm{i}\sinh(x - \mathrm{i}\theta)\partial_z], \tag{14}$$

$$\bar{D} = \bar{m}^\mu \partial_\mu = \frac{1}{\sqrt{\sinh 2x}}[\cosh(x + \mathrm{i}\theta)\partial_y - \mathrm{i}\sinh(x + \mathrm{i}\theta)\partial_z], \tag{15}$$

$$\delta = l^\mu \partial_\mu = \frac{1}{a\sqrt{\sinh 2x}}[\partial_x + \mathrm{i}\partial_\theta], \tag{16}$$

$$\bar{\delta} = \bar{l}^\mu \partial_\mu = \frac{1}{a\sqrt{\sinh 2x}}[\partial_x - \mathrm{i}\partial_\theta]. \tag{17}$$

The massive Dirac equation reads

$$\mathrm{i}\gamma^\mu \nabla_\mu \Psi = M\Psi, \tag{18}$$

where

$$\nabla_\mu = \partial_\mu - \Gamma_\mu. \tag{19}$$

The γ matrices can be written in terms of base vectors as

$$\gamma^\mu = \sqrt{2}\begin{pmatrix} 0 & 0 & l^\mu & m^\mu \\ 0 & 0 & -\bar{m}^\mu & \bar{l}^\mu \\ \bar{l}^\mu & -m^\mu & 0 & 0 \\ \bar{m}^\mu & l^\mu & 0 & 0 \end{pmatrix}. \tag{20}$$

These matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{21}$$

The spin connection is written as

$$\Gamma_\mu = \tfrac{1}{4}\gamma^\nu_{;\mu}\gamma_\nu. \tag{22}$$

In an expanded form, these equations read

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\left\{(\partial_x + \mathrm{i}\partial_\theta)\Psi_3 + a[\cos(\theta + \mathrm{i}x)\partial_y + \sin(\theta + \mathrm{i}x)\partial_z]\Psi_4 - \frac{Ma\sqrt{\sinh 2x}}{\sqrt{2}}\Psi_1\right\} = 0, \tag{23}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\left\{(\partial_x - \mathrm{i}\partial_\theta)\Psi_4 - a[\cos(\theta - \mathrm{i}x)\partial_y + \sin(\theta - \mathrm{i}x)\partial_z]\Psi_3 - \frac{Ma\sqrt{\sinh 2x}}{\sqrt{2}}\Psi_2\right\} = 0, \tag{24}$$

$$\begin{aligned}\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\Bigg\{&(\partial_x - \mathrm{i}\partial_\theta + \coth 2x\,)\Psi_1 \\ &- a[\cos(\theta + \mathrm{i}x)\partial_y + \sin(\theta + \mathrm{i}x)\partial_z]\Psi_2 - \frac{Ma\sqrt{\sinh 2x}}{\sqrt{2}}\Psi_3\Bigg\} = 0,\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\sqrt{2}}{a\sqrt{\sinh 2x}}\Bigg\{&(\partial_x + \mathrm{i}\partial_\theta + \coth 2x)\Psi_2 \\ &+ a[\cos(\theta - \mathrm{i}x)\partial_y + \sin(\theta - \mathrm{i}x)\partial_z]\Psi_1 - \frac{Ma\sqrt{\sinh 2x}}{\sqrt{2}}\Psi_4\Bigg\} = 0.\end{aligned} \tag{26}$$

To simplify calculations, we will study the massless case. Then we see that only $\{\Psi_1, \Psi_2\}$ and $\{\Psi_3, \Psi_4\}$ are coupled to each other. If we take

$$\Psi_i = \mathrm{e}^{\mathrm{i}(k_y y + k_z z)}\boldsymbol{\Psi}_{\mathbf{i}}(\mathbf{x}, \theta) \tag{27}$$

and make the transformations

$$k_y = k\cos\phi, \qquad k_z = k\sin\phi, \tag{28}$$

we get

$$\boldsymbol{\Psi}_{\mathbf{1}} = \frac{\sinh[x - \mathrm{i}(\theta - \phi)]}{\sqrt{\sinh 2x}}\Psi_1 \tag{29}$$

and

$$\boldsymbol{\Psi}_{\mathbf{2}} = \frac{\sinh[x + \mathrm{i}(\theta - \phi)]}{\sqrt{\sinh 2x}}\Psi_2. \tag{30}$$

Now we have to solve

$$L_{1,2}\Psi_{1,2} = \frac{-2}{2ak\sqrt{\sinh 2x}}\left\{\partial_{xx} + \partial_{\theta\theta} + \frac{a^2k^2}{2}\left\{\cos[2(\theta+\phi)] - \cosh 2x\right\}\right\}\Psi_{1,2} = 0, \tag{31}$$

whose solutions can be expressed in terms of Mathieu functions:

$$\Psi_{\mathbf{1}} = \mathrm{e}^{\mathrm{i}k(z\sin\phi + y\cos\phi)}\frac{\sinh[x - \mathrm{i}(\theta-\phi)]}{\sqrt{\sinh 2x}} \times\left\{\left[Se\left(\zeta_1, -\frac{a^2k^2}{4}, -\mathrm{i}x\right) + So\left(\zeta_1, -\frac{a^2k^2}{4}, -\mathrm{i}x\right)\right]\right. \times\left.\left[Se\left(\zeta_1, -\frac{a^2k^2}{4}, \theta+\phi\right) + So\left(\zeta_1, -\frac{a^2k^2}{4}, \theta+\phi\right)\right]\right\}, \tag{32}$$

$$\Psi_{\mathbf{2}} = \mathrm{e}^{\mathrm{i}k(z\sin\phi + y\cos\phi)}\frac{\sinh[x + \mathrm{i}(\theta-\phi)]}{\sqrt{\sinh 2x}} \times\left\{\left[Se\left(\zeta_2, -\frac{a^2k^2}{4}, -\mathrm{i}x\right) + So\left(\zeta_2, -\frac{a^2k^2}{4}, -\mathrm{i}x\right)\right]\right. \times\left.\left[Se\left(\zeta_2, -\frac{a^2k^2}{4}, \theta+\phi\right) + So\left(\zeta_2, -\frac{a^2k^2}{4}, \theta+\phi\right)\right]\right\}. \tag{33}$$

When similar transformations are done for the other components, we get

$$L_3\Psi_3 = \frac{\cosh[x - \mathrm{i}(\theta-\phi)]}{ak}\left\{\partial_{xx} + \partial_{\theta\theta} + \frac{a^2k^2}{2}\left\{\cos[2(\theta+\phi)] - \cosh 2x\right\}\right\}\Psi_3 = 0, \tag{34}$$

$$L_4\Psi_4 = \frac{\cosh[x + \mathrm{i}(\theta-\phi)]}{ak}\left\{\partial_{xx} + \partial_{\theta\theta} + \frac{a^2k^2}{2}\left\{\cos[2(\theta+\phi)] - \cosh 2x\right\}\right\}\Psi_4 = 0. \tag{35}$$

The solutions can again be expressed in terms of Mathieu functions:

$$\Psi_{\mathbf{3}} = \mathrm{e}^{\mathrm{i}k(z\sin\phi + y\cos\phi)}\left\{\left[Se\left(\zeta_3, -\frac{a^2k^2}{4}, -\mathrm{i}x\right) + So\left(\zeta_3, -\frac{a^2k^2}{4}, -\mathrm{i}x\right)\right]\right. \times\left.\left[Se\left(\zeta_3, -\frac{a^2k^2}{4}, \theta+\phi\right) + So\left(\zeta_3, -\frac{a^2k^2}{4}, \theta+\phi\right)\right]\right\}, \tag{36}$$

$$\Psi_{\mathbf{4}} = \mathrm{e}^{\mathrm{i}k(z\sin\phi + y\cos\phi)}\left\{\left[Se(\zeta_4, -\frac{a^2k^2}{4}, -\mathrm{i}x) + So\left(\zeta_4, -\frac{a^2k^2}{4}, -\mathrm{i}x\right)\right]\right. \times\left.\left[Se\left(\zeta_4, -\frac{a^2k^2}{4}, \theta+\phi\right) + So\left(\zeta_4, -\frac{a^2k^2}{4}, \theta+\phi\right)\right]\right\}. \tag{37}$$

Here, note that

$$\left[Se\left(\zeta_4, -\frac{a^2k^2}{4}, -\mathrm{i}x\right) + So\left(\zeta_4, -\frac{a^2k^2}{4}, -\mathrm{i}x\right)\right]$$

can be expressed in terms of modified Mathieu functions with real arguments. Here, ζ_i are separation constants.

At this point also note that we can get solutions for $\Psi_{3,4}$ in the plane wave form, which are given as $\exp(ka(\sin(\theta-\phi+\mathrm{i}x)+\sin(\theta-\phi-\mathrm{i}x)))$, similar to the ones given by Sucu and Ünal [15]. Here, since we want to point to the occurrence of Mathieu functions in mathematical physics, we use the product form. This form is also more useful when boundary conditions are imposed on the solution.

3. Equations in five dimensions

The addition of the time component to the previous metric gives

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}s_4^2, \tag{38}$$

resulting in the massless Dirac equation as

$$(\gamma^\mu \partial_\mu + \gamma^t \partial_t - \gamma^\mu \Gamma_\mu - \gamma^t \Gamma_t)\Psi = 0. \tag{39}$$

Here,

$$\Gamma_t = 0 \tag{40}$$

and

$$\gamma^t = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \tag{41}$$

giving the set of equations,

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}} \left\{ (\partial_x + \mathrm{i}\partial_\theta)\Psi_3 + a[\cos(\theta + \mathrm{i}x)\partial_y + \sin(\theta + \mathrm{i}x)\partial_z]\Psi_4 + \mathrm{i}\frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t \Psi_1 \right\} = 0, \tag{42}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}} \left\{ (\partial_x - \mathrm{i}\partial_\theta)\Psi_4 - a[\cos(\theta - \mathrm{i}x)\partial_y + \sin(\theta - \mathrm{i}x)\partial_z]\Psi_3 + \mathrm{i}\frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t \Psi_2 \right\} = 0, \tag{43}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}} \left\{ (\partial_x - \mathrm{i}\partial_\theta + \coth 2x)\Psi_1 - a[\cos(\theta + \mathrm{i}x)\partial_y + \sin(\theta + \mathrm{i}x)\partial_z]\Psi_2 - \mathrm{i}\frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t \Psi_3 \right\} = 0, \tag{44}$$

$$\frac{\sqrt{2}}{a\sqrt{\sinh 2x}} \left\{ (\partial_x + \mathrm{i}\partial_\theta + \coth 2x)\Psi_2 + a[\cos(\theta - \mathrm{i}x)\partial_y + \sin(\theta - \mathrm{i}x)\partial_z]\Psi_1 - \mathrm{i}\frac{a\sqrt{\sinh 2x}}{\sqrt{2}}\partial_t \Psi_4 \right\} = 0. \tag{45}$$

If we solve for Ψ_1 and Ψ_2 and replace them in the latter equations, we get two equations which has only Ψ_3 and Ψ_4 in them. If we take

$$\Psi_i = \mathrm{e}^{\mathrm{i}(k_t t + k_y y + k_z z)} \Psi_{\mathbf{i}}(x, \theta), \tag{46}$$

the resulting equations read

$$\left\{ \partial_{xx} + \partial_{\theta\theta} + \frac{a^2 k^2}{2} \{\cos[2(\theta + \phi)] - \cosh 2x\} + 2a^2 k_t^2 \sinh 2x \right\} \Psi_{3,4} = 0. \tag{47}$$

If we assume that the result is expressed in the product form $\Psi_3 = T_1(x)T_2(\theta)$, the angular part is again expressible in terms of Mathieu functions:

$$T_2(\theta) = Se\left[\eta, -\frac{a^2k^2}{4}, \arccos\left(\sqrt{\frac{1+\cos(\theta+\phi)}{2}}\right)\right] + So\left[\eta, -\frac{a^2k^2}{4}, \arccos\left(\sqrt{\frac{1+\cos(\theta+\phi)}{2}}\right)\right]. \tag{48}$$

Here, η is the separation constant and the periodicity on the solution makes it equal to the square of an integer. We are interested only in periodic solutions of this equation with period 2π. These solutions exist only for discrete values of the separation constant η and they are given by even and odd periodic Mathieu functions $Se_n(ka, \cos\Theta)$ and $So_n(ka, \cos\Theta)$, respectively. When the parameter ka tends to zero, these solutions reduce to the trigonometric functions

$$Se_n(ka, \cos\Theta) \to \cos(n\Theta), \qquad So_n(ka, \cos\Theta) \to \sin(n\Theta)$$

while the separation constant

$$\eta(e_n) \to \eta(o_n) \to n^2$$

where n^2 is square of an integer [18, 35]. The equation for T_1 reads

$$\left\{\partial_{xx} - \frac{a^2k^2}{2}\cosh 2x + 2a^2k_t^2 \sinh 2x - \eta\right\} T_1 = 0. \tag{49}$$

The solution of this equation is expressed in terms of double confluent Heun functions [4]:

$$T_1(x) = HeunD\left[0, \frac{a^2k^2}{2} + \eta, 4a^2k_t^2, \frac{a^2k^2}{2} - \eta, \tanh x\right] + HeunD\left[0, \eta + \frac{a^2k^2}{2}, 4a^2k_t^2, \frac{a^2k^2}{2} - \eta, \tanh x\right] \times \int \frac{-\mathrm{d}x}{HeunD\left[0, \eta + \frac{a^2k^2}{2}, 4a^2k_t^2, \frac{a^2k^2}{2} - \eta, \tanh x\right]^2}. \tag{50}$$

We only take the first function and discard the second solution. We see that as x goes to infinity, the function given above diverges. The function is finite at $x = 0$ though. In order to get well-defined functions, we study the region where $x \leqslant F$, where F is a finite value. We will give a way to determine F below.

We can use either Dirichlet or Neumann boundary conditions for our problem in four dimensions. There is an obstruction in odd Euclidean dimensions that makes us use the Atiyah–Patodi–Singer [33] spectral boundary conditions. These boundary conditions can also be used in even Euclidean dimensions if we want to respect the charge conjugation and the γ^5 symmetry [34]. Just to show the differences with the four-dimensional solution, we attempt to write this expression in terms of Mathieu functions. This can be done after few transformations. We define

$$A = 2a^2k_t^2, \tag{51}$$

$$B = -\eta, \tag{52}$$

$$C = -\frac{a^2k^2}{2}, \tag{53}$$

and use the transformation

$$z = \mathrm{e}^{-2x}. \tag{54}$$

Then the differential operator is expressed as

$$O = 4z^2\partial_{zz} + 4z\partial_z + A'z + B + C'\frac{1}{z}. \tag{55}$$

Here

$$A' = \frac{C - A}{2}, \tag{56}$$

$$C' = \frac{C + A}{2}. \tag{57}$$

If we take

$$\sqrt{\frac{C'}{A'}}u = z, \tag{58}$$

and

$$w = \frac{1}{2}\left(u + \frac{1}{u}\right) \tag{59}$$

and set $E = \sqrt{A'C'}$, we get

$$O = (w^2 - 1)\partial_{ww} + w\partial_w + \frac{E}{2}w + \frac{B}{4}. \tag{60}$$

The solution of this equation is also expressible in terms of Mathieu functions given as

$$R(z) = Se(-B, E, \arccos\sqrt{w + 1}) + So(-B, E, \arccos\sqrt{w + 1}). \tag{61}$$

At this point we see a natural limitation in the values that can be taken by our radial variable since the argument of the function arccos cannot exceed unity. The fact that $\sqrt{w + 1}$ cannot exceed unity limits the values our initial variable x can take, thus determining F which imposed on our solution in equation [50].

Here we also see a difference from the four-dimensional case. Although both the radial and the angular parts can be written in terms of Mathieu functions, the constants are different, modified by the presence of the new $-2a^2k_{t^2}^2$ term, which makes the summation of these functions to form the propagator quite difficult.

In four dimensions we can use the summation formula [18, 35] for the product of four Mathieu functions, two of them for the angular part and the other two for the radial part, summing them to give us a Bessel-type expression. This result makes the calculation of the propagator, similar to the case given in [18], possible. In that case, the similar analysis also gives the solution to the differential equation in a closed form as given in [15]. Here since the radial and angular parts have different constants, this summation formula is not applicable to write the Greens function in a closed form. We also see that using the generating function formula for these functions to write the solution to the differential equation in terms of plane waves, as described in the paper by L Chaos-Cador *et al* is not applicable in the five-dimensional case due to the same reason.

4. Laplacian

In this section we give the Laplacian operator written in this background. It is used for the calculation of the field equation for the scalar particle, similar to the case studied in [18]:

$$H := \frac{1}{\sqrt{-g}}\partial_\nu\sqrt{-g}g^{\mu\nu}\partial_\mu \tag{62}$$

$$H := -\partial_{tt} + \partial_{xx} + \partial_{\theta\theta} + a^2 \sinh^2 2x(\partial_{yy} + \partial_{zz}) \\ + a^2(\cos\theta\,\partial_y + \sin\theta\,\partial_z)^2 - a^2 \sinh x \cosh x\,\partial_{tt}. \tag{63}$$

We see that there are three Killing vectors and one quadratic Killing tensor with eigenvalues given below. From the Killing tensor, we can construct a second-order operator tensor [18]

$$K = -\partial_{\theta\theta} - a^2(\cos\theta\,\partial_y + \sin\theta\,\partial_z)^2 \tag{64}$$

$$K\Phi = \lambda\Phi. \tag{65}$$

We use λ as the separation constant. The other eigenvalues are

$$\partial_t\Phi = k_t\Phi, \tag{66}$$

$$\partial_y\Phi = k_y\Phi, \tag{67}$$

$$\partial_z\Phi = k_z\Phi. \tag{68}$$

We have

$$\Phi = \mathrm{e}^{\mathrm{i}(k_t t + k_y y + k_z z)} R(x) S(\theta), \tag{69}$$

where S obeys the equation

$$\frac{\mathrm{d}^2\tilde{S}}{\mathrm{d}\tilde{\theta}^2} + (\lambda - a^2k^2\cos^2\tilde{\theta})\tilde{S} = 0. \tag{70}$$

Here

$$\tilde{\theta} = \theta - \phi, \tag{71}$$

and the solution reads

$$S(\theta) = Se\left(\frac{-a^2k^2}{2} + \lambda, \frac{a^2k^2}{4}, \theta - \phi\right) + So\left(\frac{-a^2k^2}{2} + \lambda, \frac{a^2k^2}{4}, \theta - \phi\right). \tag{72}$$

λ is the separation constant which goes to the square of an integer due to periodicity of the function. The radial part obeys

$$\frac{\mathrm{d}^2R}{\mathrm{d}x^2} + a^2\left(\sinh x \cosh x k_t^2 - k^2\sinh^2 x - \frac{\lambda}{a^2}\right) R = 0. \tag{73}$$

The solution to this equation, the radial solution, can be written in terms of double confluent Heun functions:

$$R(x) = HeunD\left[0, a^2k^2 - \lambda, -a^2k_t^2, \lambda, \tanh x\right] + HeunD\left[0, a^2k^2 - \lambda, -a^2k_t^2, \lambda, \tanh x\right] \\ \times \int \frac{-\mathrm{d}x}{HeunD\left[0, a^2k^2 - \lambda, -a^2k_t^2, \lambda, \tanh x\right]^2}, \tag{74}$$

and this can be reduced to the modified Mathieu function after performing similar transformation as in the spinor case treated above. Taking $A = \frac{a^2k_t^2}{2}$, $B = \frac{a^2k^2}{2} - \lambda$ and $C = \frac{-a^2k^2}{2}$, and using $z = \mathrm{e}^{-2x}$ transformation we get the same result as in equation (61).

5. Conclusion

Here we related solutions of the Dirac equation in the background of the Nutku helicoid solution in five dimensions to the double confluent Heun function. This solution can be also expressed in terms of the Mathieu function, which is more familiar to the physics community, at the expense of using the $z = e^{-2x}$ transformation, which maps infinity to zero followed by a rescaling and a further transformation where, aside from the scaling, we are taking the hyperbolic cosine of the original radial variable. This transformation also brings a natural limit on the radial variable. Essentially we do not gain much, since the Mathieu function also has a three way recursion relation, and a not very handy generating function, compared to the generating functions belonging to the hypergeometric family. Such a transformation to Mathieu functions may not be possible when more complicated backgrounds are taken. Although in many cases the Heun function may reduce to more simple forms [36, 37], there are metrics in general relativity where this may not be possible. The Nutku helicoid metric is such an example.

We know that using Maple 10 program we can see the graphical representation of these functions. Different asymptotic solutions are also studied in [3–5] and in individual papers. Still it is our feeling that most of the mathematical physics community is not at ease with these functions as they are with the better-known hypergeometric or confluent hypergeometric functions. The literature on these functions is also very limited. We think we have covered most of the monographs where these functions are thoroughly studied in our bibliography. If we search the SPIRES website, which is commonly used by mathematical physicists, the number of entities is rather small. This is the reason we think one should be exposed to its applications more often.

We also wanted to show in our work that when one uses more complicated forms of similar structures, one gets more complicated solutions. What we mean by this phrase is that if one writes the Dirac equation in the background of the simplest Euclidean solution in four dimensions, the Eguchi–Hanson solution, one gets the hypergeometric function as solutions of the wave equation [15]. If we use the next solution in the order of complexity, the Nutku helicoid solution, one gets a solution of the Heun type. We think the solution obtained in [15] for this case in terms of exponential functions is somewhat misleading, since there they get the generating function of the Mathieu functions. This may not be recognized by the readers unless one is an expert in this field.

We also studied the same equation in one higher dimension. Often, increasing the number of dimensions of the manifold in which the wave equation is written results in higher functions as solutions. Here we call a function of a higher type if it has more singularities. In this respect, the Heun function belongs to a higher form than the hypergeometric function. In the introduction, we gave examples of the use of Heun functions encountered in different physics problems.

In the future we will try to find further examples of such functions encountered in solutions to wave equations in general relativity.

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